PROBLEMS FROM THE HISTORY OF MATHEMATICS
PROBLEM SET #9

DUE FRIDAY, 4/20/2018

Exercise 1. In this problem we prove Thue's Theorem, ie. that the regular hexagonal lattice (in two dimensions) achieves an optimal packing density. Throughout, let Λ be the set of centers of a saturated packing of unit circles in the plane. It may be helpful to refer to the Figure below throughout.

a. Fix a triangle Δ in a Delauney triangulation of Λ and let θ be the largest angle in Δ. Show that $\theta \geq \frac{2\pi}{3}$ implies that the circumradius of Δ is ≥ 2. *Hint: Law of Sines.*
b. Assume that $\theta \geq \frac{2\pi}{3}$. Show that the circumcenter of Δ is at least 2 units away from every point in Λ. Why is this a contradiction? Conclude that $\frac{\pi}{3} \leq \theta < \frac{2\pi}{3}$.
c. Prove that the area of Δ is at least $\sqrt{3}$.
d. What proportion of the area of the triangle Δ is covered by the unit circles at its vertices? (Your answer will depend on the area of Δ.)
e. Prove that the optimal packing density in the plane is $\pi/\sqrt{12}$.

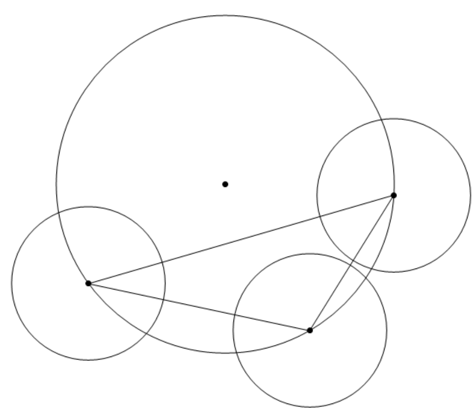

FIGURE 1. A decidedly obtuse Delauney triangle.

Exercise 2. This exercise will produce an elementary lower bound on $\pi(x)$ which complements the upper bound we established in class. Again, we rely on Legendre's theorem.

a. Prove that

$$n \log 4 \geq \log \binom{2n}{n} \geq \sum_{p \leq 2n} \left(\left\lfloor \frac{2n}{p} \right\rfloor - 2 \left\lfloor \frac{n}{p} \right\rfloor \right) \log p.$$

b. Recall that $\vartheta(x) = \sum_{p \leq x} \log p$, in which the sum is extended only over primes. Prove that

$$n \log 4 \geq \sum_{n < p \leq 2n} \log p = \vartheta(2n) - \vartheta(n).$$

c. Apply (b) to show $\vartheta(n) \geq n \log 4$. Use this to prove that

$$\pi(n) \geq \frac{n \log 4}{\log n}.$$

Exercise 3. Prove that the following are equivalent:

a. $\pi(x) \sim x / \log x$
b. $\vartheta(x) \sim x$
c. $\psi(x) \sim x$.