PROBLEMS FROM THE HISTORY OF MATHEMATICS
PROBLEM SET #8

DUE FRIDAY, 4/13/2018

Exercise 1. Recall the following corollary to Liouville's theorem in differential algebra:

Corollary. *Choose* $f, g \in \mathbb{C}(x)$ *with* $f \neq 0$ *and* g *non-constant. Then* $f(x)e^{g(x)}$ *can be integrated in elementary terms if and only if there exists a* ***rational*** *function* $R(x) \in \mathbb{C}(x)$ *such that* $R'(x) + g'(x)R(x) = f(x)$.

a. Prove that e^x/x has no elementary anti-derivative.
b. Prove that e^{-x^2} has no elementary anti-derivative.

Exercise 2. In this exercise, we derive a rapidly converging series for $\zeta(2)$ due to Euler (1731).

a. Prove that
$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t} dt.$$
b. Fix $\alpha \in (0,1)$. Prove that
$$-\int_\alpha^1 \frac{\log(1-t)}{t} dt = \log\alpha \log(1-\alpha) - \int_0^{1-\alpha} \frac{\log(1-t)}{t} dt.$$
c. Prove that
$$\zeta(2) = (\log 2)^2 + \sum_{n=1}^{\infty} \frac{2^{1-n}}{n^2}.$$

Exercise 3. Adapt Euler's proof that $\zeta(2) = \frac{\pi^2}{6}$ to find a closed form (involving only powers of π and rational numbers) for $\zeta(4)$.

Exercise 4. Find an Eulerian path (or prove that one does not exist) in the graph presented in Figure 1.

Exercise 5. A *Hamiltonian path* is a path between two vertices in a graph that visits each vertex exactly once. Compute the number of Hamiltonian paths in the complete graph K_n on n vertices.[1]

[1] K_n is the graph on n vertices which includes an edge between any vertex pair.

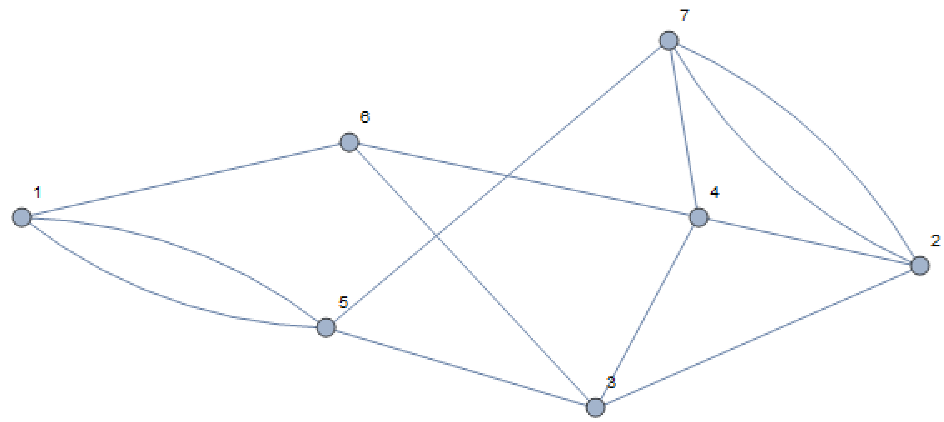

FIGURE 1. The graph for Exercise 4.

Exercise 6. Describe a map on the surface of a torus which cannot be colored using 4 colors. Can you find one that requires 6 or even 7 colors? *Hint: It may help to visualize the torus as a square with opposite edges identified.*

Exercise 7. Visit the website www.rotopo.com and play the demo version of the game you find there. Continue playing until you recognize that finding Hamiltonian paths is not always trivial.[2]

[2]This isn't a real exercise but you should check out the site anyway!