Hilbert's 23 Problems

Problems from the History of Mathematics

Lecture 22 — April 18, 2018

Brown University

Hilbert's 23 Problems

David Hilbert (1862-1943) was a German mathematician who played a major role in

1. the axiomatization of mathematics[1]
2. the development of functional analysis (Hilbert spaces, etc.)
3. the development of metamathematics

On August 8, 1900, Hilbert presented a list of ten problems to the Paris conference of the International Congress of Mathematicians. This list was expanded to 23 problems before publication in 1902.[2]

Many of these problems have been influential in the development of math in the last century.

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[1] We have already mentioned Hilbert's re-axiomatization of the *Elements*.
[2] *Mathematical Problems*, Bulletin of the American Mathematical Society, 1902.

H1: Cantor's Continuum Hypothesis

Two sets A and B are said to have the same cardinality if there exists a bijection between them. For finite sets, cardinality simply captures the number of elements in the set.

Infinite sets are more involved. A set is called countable if it has the same cardinality as the integers.

Theorem (Cantor, c. 1874-1884):

1. The rational numbers are countable.
2. The real numbers are not countable.[3]

Sets in bijection with $\mathbb{R}$ have the cardinality of the continuum.

The Continuum Hypothesis (Cantor):

There exist no cardinalities between that of $\mathbb{N}$ and $\mathbb{R}$.

[3] Cantor's famous diagonalization argument (1891). Other proofs show that a set is never in bijection with its power set (set of subsets).

Status of H1

The continuum hypothesis was resolved by the combined work of Gödel and Cohen:

1. In 1940, Gödel showed that the Continuum Hypothesis could not be disproved within Zermelo–Fraenkel set theory (ZF).
2. In 1964, Cohen showed that the Continuum Hypothesis could not be proven within ZF. Cohen developed the now-standard method of forcing to prove this result.

CH was one of the first results to be proven independent of ZFC.

H2: Consistency of Axiomatic Arithmetic

Axiomatization of modern mathematics took off in the late 1800s. Given an axiomatic system, there are two natural questions:

1. Are the axioms independent of each other?
2. Does the theory generated by these axioms contain contradictory results?

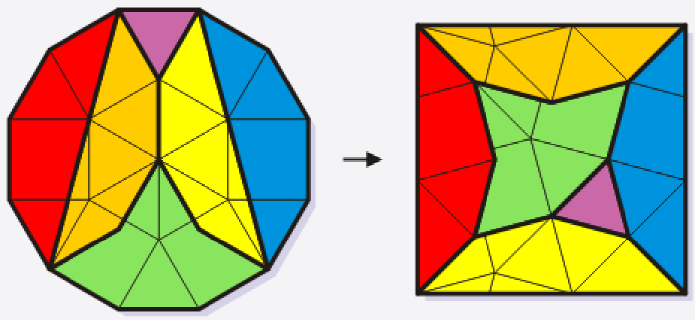
In 1931, Gödel's incompleteness theorem showed that no proof of the consistency of arithmetic could be carried out in arithmetic itself. Bolstering arithmetic by adding more axioms leads to more statements which, again, cannot be proven within the new system.

Hilbert's question "prove that the axioms of arithmetic are consistent" is too vague to answer definitely, but Gödel's work shows that the proving part would have to take place elsewhere.

Dissection of Polyhedra

In two dimensions, the Wallace–Bolyai–Gerwein theorem implies that any two polygons of equal area can be divided into (finitely many) polygonal pieces and reassembled into each other.

For example, we may dissect a dodecagon into a square:



Status of H3

Hilbert wondered if a similar result should hold for tetrahedra of equal volume. If so, then the volume formula for a tetrahedra could be developed without appeal to the method of exhaustion (or similar).

In 1901, Max Dehn proved that this was not possible by developing the theory of Dehn invariants. The Dehn ivariant of a polyhedron with edge lengths ℓ_i and dihedral angles[4] θ_i is the tensor product

$$\sum_i \ell_i \otimes_{\mathbb{Z}} \theta_i,$$

Dehn proved that dissection preserved the Dehn invariant (and that two solids can have different Dehn invariants). In 1965, Snyder showed that volume and Dehn invariant are the only invariants for dissection.

[4] The angle between intersecting planes.

H4: Construct all Metrics in which Lines are Geodesics

In the pursuit of non-Euclidean geometries several mathematicians attempted to define a straight line as the shortest distance between two points. Equivalently, these are geometries with a triangle inequality.

Hilbert was motivated to ask this question based on Minkowski's work on the geometry of spacetime (called Minkowski space), in which one has a reversed triangle inequality.

H4, in its original formulation, has been deemed too vague to ever be completely resolved. The classification for convex subsets of the plane was resolved by Ambartzumian in 1976.

H5: Are continuous groups necessarily differential groups?

A Lie group is a group that is also a differentiable manifold, in such a way that the group actions are compatible with the smooth structure.

Hilbert asks:

> *How far Lie's concept of continuous groups of transformations is approachable in our investigations without the assumption of the differentiability of the functions.*

Hilbert's question is slightly vague (especially since he did not view manifolds as we do). Some people view this problem as solved – differentiability is not required; continuity is enough – with a resolution in 1953 by Gleason–Montgomery–Zippin and Yamabe.

A different reading of H5 leads to the Hilbert–Smith Conjecture, which is currently open.

H6: Mathematical Treatment of the Axioms of Physics

Hilbert's 6th problem concerns not mathematics, but the mathematical sciences:

> *To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.*

Roughly, H6 concerns the axiomatization of fields beyond mathematics. In some cases this has been done successfully: Kolmogorov axiomatized probability theory with measure theory in the 1930s.

General relativity (1910) and quantum field theory (1920-1960?) attempt to axiomatize physics, but are not even compatible with each other.

As a question, H6 is too general to be resolved anytime soon.

H7: Irrationality and Transcendence of Certain Numbers

Hilbert asks two specific questions:

1. If the ratio of two angles in an isosceles triangle is algebraic and irrational, is the ratio of base and side lengths always transcendental?
2. If a^b always transcendental, for algebraic $a \notin 0, 1$ and b an algebriac irrational?

These two questions are essentially equivalent. H7, in the second form, was answered in the affirmative by Gelfond (1934) and Schneider (1935). Their result is now known as the Gelfond–Schneider Theorem.[5]

[5] This program has been generalized in Baker's work on linear forms in logarithms.

H8: Distribution of Primes

Now onto number theory, Hilbert includes several problems which concern the distribution of primes. These include:

1. The (Generalized) Riemann Hypothesis
2. Goldbach's Conjecture
3. The Twin Prime Conjecture

Each remains open. Partial progress has been made on Goldbach's Conjecture[6] and the Twin Prime Conjecture.[7]

In contrast, little substantive progress has been made towards GRH.

[6] Weak Goldbach: Every odd number ≥ 7 is the sum of three primes. (Helfgott, 2013)
[7] There exist prime gaps of length ≤ 246 which occur infinitely often. (Zhang, Maynard, Tao, Polymath8, 2013)

H9: Reciprocity Laws

The law of quadratic reciprocity gives conditions which link the solvability of the equations

$$x^2 \equiv p \mod q \quad \text{and} \quad y^2 \equiv q \mod p$$

to each other. It was first proven by Gauss in *Disquisitiones Arithmeticae*.

Hilbert asks for generalizations of these laws beyond quadratic symbols and to any number field. This would be be partially resolved by Artin (1927) in the case of abelian extensions by the development of class field theory.

Non-abelian class field theory is harder, so H9 is only partially resolved.

H10: Algorithms for Solving Diophantine Equations

Hilbert writes:

> *"Given a diophantine equation with any number of unknown quantities and with rational integral numerical coefficients: To devise a process according to which it can be determined by a finite number of operations whether the equation is solvable in rational integers."*

H10 was proven to be impossible by Matiyasevich's theorem in 1970. Roughly, one proves that recursively enumerable sets are exactly Diophantine sets. But there exist sets which are recursively enumerable but not computable, so no general algorithm could exist.

H11: Quadratic Forms over Number Fields

A quadratic form is a homogeneous polynomial of degree two, such as

$$x^2 + 2y^2 + xz.$$

Hilbert notes that the theory of quadratic forms has been well developed when the coefficients lie in $\mathbb{Q}$. H11 deals with an extension of this theory to quadratic forms with algebraic coefficients.

Hilbert does not clarify which problems should be further developed. Reasonable interpretations include

1. representability by a given quadratic form
2. equivalence classes of quadratic forms (partially settled by the Hasse–Minkowski theorem)

H11 is probably too vague to ever be considered completely solved.

H12: Extend the Kronecker–Weber Theorem

Like many of Hilbert's questions in number theory, H12 concerns the generalization of a result about $\mathbb{Q}$ to the case of number fields. Here, the base result is the Kronecker–Weber Theorem:

Theorem (Kronecker–Weber–Hilbert, 1853-1886-1896):

Every finite abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field.

Roughly speaking, H12 asks: which algebraic numbers are required to construct all abelian extensions of a given number field k?[8]

H12 has been solved only when k is an imaginary quadratic field or a CM-field.[9]

Whereas one moves from $\mathbb{Q}$ to abelian extensions by adjoining special values of $e^{2\pi i z}$, the solution for CM-fields involves special values of the lattice-based functions $\wp(\tau, z)$ and $j(\tau)$.

[8] These numbers take the place of the roots of unity.

[9] A CM-field is a totally imaginary field that is a quadratic extension of a totally real field.

H13: Solving the Septic using Functions of Two Variables

Hilbert remarks that the roots of any equation which is solvable by radicals may be obtained by composing functions of two variables.[10]

The general quintic can be solved if we allow ourselves use of the inverse function of $x = y^5 + y$. Thus the quintic may be solved by composing algebraic functions of two variables.

H13 asks if the same should hold for the septic (the sextic was known).

H13 is partially resolved. Vladimir Arnold, then a student of Kolmogorov, showed in 1957 that this was possible if the functions of two variables were only assumed to be continuous (as opposed to algebraic).

[10] These functions are addition, multiplication, division, and root extraction.

H14: Finitely Generated Algebras

For H14, let k be a field and let K be a subfield of the field of rational functions in n variables over k, ie $K \subset k(x_1, \ldots, x_n)$.

H14 Conjecture:

The k-algebra defined by $K \cap k[x_1, \ldots, x_n]$ is finitely generated over k.

This conjecture was proven for $n = 1$ and $n = 2$ by Zariski in 1954. In 1959, Masayoshi Nagata found a counterexample to Hilbert's conjecture.

The counterexample draws on the theory of linear algebraic groups, which are subgroups of the general linear group $\mathrm{GL}(n, \mathbb{C})$.

H15: Foundations of Schubert's Enumerative Calculus

Intersection theory describes the intersections of algebraic varieties. Unfortunately, naive combinatorics can sometimes fail to give accurate numbers for the dimensions of intersected manifolds.

In the 19th century, Schubert developed a calculus[11] for determining intersections. H15 asks for a rigorous foundation for this theory, which was eventually given in terms of Grassmanians (and later work in intersection theory).

H15 also asks for a rigorous foundation of other tools in enumerative geometry. The status of this form of H15 depends on interpretation.

[11] This is 'calculus' in the traditional sense of 'computational aid.'

H16: Connected Components of Algebraic Curves

Hilbert begins H16 by referencing a recent work of Harnack:

Harnack's Curve Theorem (Harnack, 1876):

An algebraic curve in the real projective plane of degree n has at most $\frac{n^2-3n+4}{2}$ connected components.

For example, elliptic curves (degree 3) over the reals have at most 2 components.

Curves exist that attain the upper bound are known as M-curves. Hilbert believed that the 11 components of the degree 6 must lie in restricted configurations. H16 asks for a classification of these configurations. This remains open even for $n = 8$.

H16 also asks a somewhat unrelated question about the limit cycles of polynomial vector fields. The number of such cycles is known to be finite but no uniform bound is known.

H17: Rational Functions as Sums of Squares

Hilbert notes that not every non-negative polynomial can be written as a sum of squares of other polynomials.[12] For example,

$$f(x, y, z) = z^6 + x^4y^2 + x^2y^4 - 3x^2y^2z^2$$

is non-negative but is not a sum of squares.[13] H17 weakens the question:

Question (Hilbert, 1902):

Is every non-negative polynomial a sum of squares of rational functions?

Emil Artin answered H17 in the affirmative in 1927. The proof is non-constructive and applies Artin–Schreier theory. An algorithmic solution was found by Charles Delzell in 1984, which builds on Polya's theorem.[14]

[12] This was Minkowski's thesis. Hilbert was an official opponent.

[13] Explicit examples like this were first constructed by Motzkin in 1967.

[14] Let $p(x_1, \ldots, x_n)$ be homogeneous. Then $(x_1 + \ldots x_n)^N p(x_1, \ldots, x_n)$ has positive coefficients for some $N > 0$.

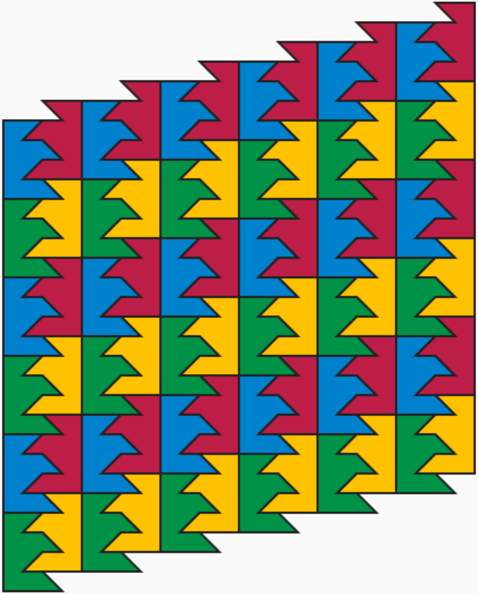
H18: Packing Space with Congruent Figures

H18 poses three different questions about packings and symmetry groups in Euclidean space:

1. Are there finitely many different space groups in n-dimensional Euclidean space? Yes, due to Bieberbach in 1911.
2. Does there exist an anisohedral polyhedral tiling of 3-dimensional Euclidean space? Yes, due to Reinhardt in 1928.[15]
3. What is the densest packing of a given solid (e.g. ball) in space? This is generally taken as Kepler's Conjecture, which was settled by Hales in 1998.

[15] Also true in the plane, due to Heesch in 1935.

H18: Heesch's Anisohedral Tiling of the Plane



H19: Analyticity of Solutions in the Calculus of Variations

Hilbert remarks his interest in the fact that there exist classes of PDEs which admit only analytic solutions. The most famous of these is the Laplace equation, but Hilbert calls out other problems of the form

$$\iint F(p,q,z;x,y)dxdy = \text{minimum}; \qquad F_{pp}\cdot F_{qq} - F_{pq}^2 > 0,$$

which arise in other minimization problems.

H19 asks whether or not the solution to these problems are necessarily analytic, even when they may only be continuous on a boundary.

Early work showed this to be the case under mild continuity assumptions (eg. C^3 solutions). The full resolution of H19, in the affirmative, is due to Ennio De Giorgi (1956-7) and John Nash (1957-8).

H20: Variational Problems with Boundary Conditions

Hilbert writes,

> *An important problem closely connected with the foregoing [H19] is the question concerning the existence of solutions of partial differential equations when the values on the boundary of the region are prescribed.*

Here, Hilbert refers to a type of PDE known as a boundary problem. Hilbert mentions partial success by Schwarz, Neumannmm and Poincaré, but mentions an interest in problems in which

1. The boundary conditions may depend on the derivatives of the solution
2. The differential equation is defined in a non-Euclidean space

H21: Differential Equations with Prescribed Monodromy

In H21, Hilbert makes the claim that

> *There always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group.*

Let's unpack this:

- A differential equation is in the Fuchsian class if its solutions involve at most first order poles.
- The monodromy group of a differential equation is the group of permutations of solutions as those solutions are analytically continued around poles.

A proof was claimed to H21 by Plemelj in 1908, which was accepted until the mid-1960s. A concrete counterexample to Plemelj's work (and H21) was found in 1990. If the Fuchsian condition is dropped, the monodromy group can be prescribed. In this sense, H21 would hold.

H22: Uniformization via Automorphic Functions

In H22, Hilbert considers surfaces which are defined by expressions of the form $F(x, y) = 0$, in which F is an analytic function of two variables. Specifically, Hilbert considers the problem of uniformization, in which surfaces may be shown to be diffeomorphic to canonical examples.

This was accomplished by Poincaré in the 19th century in the case when $F(x, y)$ was a polynomial. H22 asks for a generalization of this result to more general F.

In particular, H22 asks if the uniformization may be carried out with automorphic functions. This follows from the Uniformization Theorem of Koebe and Poincaré when $F(x, y)$ is a function of two variables.

The general case, when F is a function of more than two variables, remains open.

H23: Calculus of Variations

The problem that Hilbert lays out in H23 is deliberately vague: *further develop the theory of calculus of variations*.

This theory, which concerns minimizing functionals, has seen plenty of development over the last century.[16] Of course, H23 itself is too vague to ever be considered 'resolved.'

[16] By Hilbert, Lebesgue, and Hadamard, for example.

Questions?