The Prime Number Theorem and the Riemann Hypothesis

Problems from the History of Mathematics

Lecture 21 — April 13, 2018

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Early Bounds on the Distribution of Primes

Let $\pi(x)$ denote the number of primes at most x. In Proposition IX.20 of *Elements*, Euclid proves that

Theorem (Euclid):

We have $\pi(x) \to \infty$ as $x \to \infty$.

Euclid's method can be used to prove an explicit lower bound on $\pi(x)$. If p_k denotes the kth prime, then $p_k \leq 2^{2^{k-1}}$, hence $\pi(x) \geq \log \log x$.

On the other hand, not all numbers are prime, so $\pi(x) < x$.

Euler's Proof of the Infinitude of the Primes

Euler gave a second proof of the infinitude of the primes by considering the divergence of the harmonic series. A version of his proof follows:

For any $\sigma > 1$, we have

$$\sum_{n\geq 1} \frac{1}{n^\sigma} = \prod_{p \text{ prime}} \sum_{k\geq 0} \frac{1}{p^{k\sigma}} = \prod_p (1 - p^{-\sigma})^{-1}.$$

If the number of primes is finite, the right-hand side converges as $\sigma \to 1$. But the left-hand side diverges as $\sigma \to 1$, a contradiction. □

Corollary:

The harmonic series of primes diverges.

Proof: Take logarithms of the infinite product to turn it into an infinite sum. Approximate these logarithms to first order. □

Conjectures for the Asymptotics of $\pi(x)$

Based on extensive tables of prime counts, Legendre conjectured around 1797 that $\pi(x)$ was asymptotic to $x/\log x$.[1]

In 1838, Dirichlet conjectured that $\pi(x)$ was well-approximated by the logarithmic integral

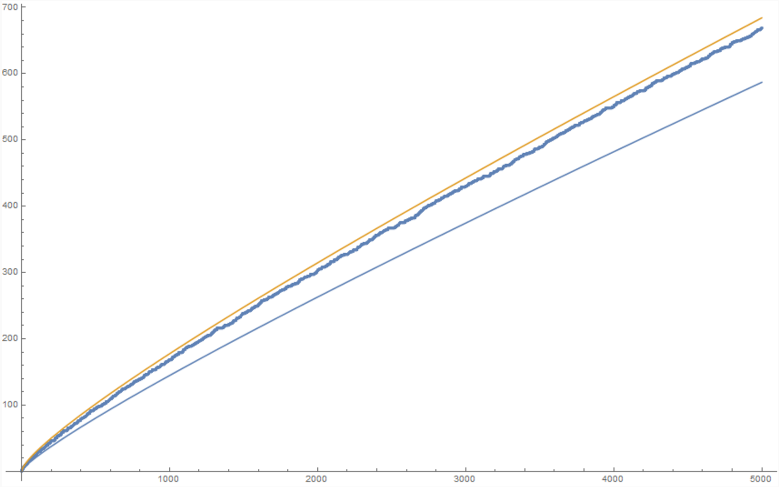
$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} = \frac{t}{\log t} + O\left(\frac{t}{(\log t)^2}\right).$$

Note that this formulation has a probabilistic interpretation: it is the expected number of primes if the probability that X is prime is $1/\log X$.

[1] Gauss claimed in 1846 to have reached the same conclusion in 1792-93.

A Comparison of Asymptotics

The following graph shows $\pi(x)$ alongside the asymptotics $x/\log x$ (blue) and $\mathrm{Li}(x)$ (orange). Note that $\mathrm{Li}(x)$ appears to offer a better fit.



Chebyshev and Riemann

Bertrand's Postulate

In 1845, Bertrand conjectured that there always existed a prime in the interval $[n, 2n]$. This conjecture, known as Bertrand's Postulate, was proven by Chebyshev in 1852.

For a morally similar result, let's prove a lower bound on $\pi(x)$.

Proof: Legendre's theorem and the bound $4^n/2n \leq \binom{2n}{n}$ give

$$n \log 4 - \log(2n) \leq \log \binom{2n}{n} = \sum_{p \leq 2n} \sum_{k \leq \left\lfloor \frac{\log 2n}{\log p} \right\rfloor} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right) \log p.$$

The parenthetical is at most 1, so

$$n \log 4 - \log(2n) \leq \sum_{p \leq 2n} \log(2n) = \pi(2n) \log(2n).$$

$$\implies \pi(n) \geq \frac{n \log 2}{\log n} - 1.$$

One may prove upper bounds for $\pi(n)$ in a similar way.

The Chebyshev Functions

The proof on the previous slide is essentially due to Erdős. Chebyshev's original proof began by relating $\pi(x)$ to the Chebyshev functions

$$\vartheta(x) = \sum_{p \le x} \log p; \qquad \psi(x) = \sum_{p^k \le x} \log p.$$

The relation $\pi(x) \sim \frac{x}{\log x}$ is equivalent to either $\vartheta(x) \sim x$ or $\psi(x) \sim x$, but the latter are more amenable to study. By proving (sufficiently sharp) bounds for $\vartheta(x)$, Chebyshev could show that $\vartheta(2x) - \vartheta(x)$ was positive.

Chebyshev also proved that if $\pi(x) \sim \frac{Ax}{\log x}$, then $A = 1$.[2]

[2] This observation can also be seen from Euler's work with the harmonic series.

The Contributions of Riemann

In 1859, Bernhard Riemann published a ten page paper which developed the connection between $\pi(x)$ and $\zeta(s)$ suggested by Euler's work.

1. $\zeta(s) = \sum_{n \geq 1} 1/n^s$ is considered as a function of a complex variable
2. $\zeta(s)$ is meromorphic and has a pole at $s = 1$ with residue 1
3. $\zeta(s)$ has an analytic continuation to all of $\mathbb{C}$
4. $\zeta(s)$ satisfies a functional equation,

$$\Lambda(s) := \pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \Lambda(1-s),$$

relating values with $\mathrm{Re}s < 0$ to those with $\mathrm{Re}s > 1$.

The Riemann Zeta Function

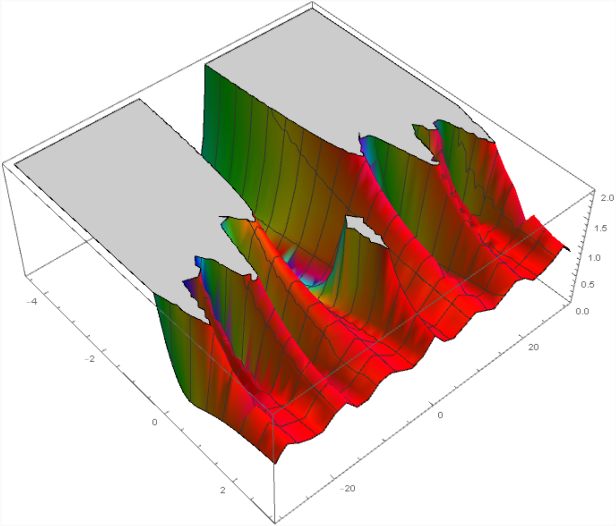


Figure 1: A magnitude-argument plot of $\zeta(s)$

The Riemann Hypothesis

Riemann uses the symmetry of $\zeta(s)$ to construct a new function $\xi(t)$ which is real on the critical line $\frac{1}{2} + it$ and has the same zeros as ζ. Riemann shows that the number of roots of $\zeta(s)$ in the critical strip $[0, 1] \times [0, T]$ is approximately $T/(2\pi) \log T$. In terms of $\xi(t)$, this corresponds to the region $\operatorname{Im}(t) \in [-\frac{1}{2}, \frac{1}{2}]$, with $t \in [0, T]$.

Riemann then writes (about $\xi(t)$):

> *One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.*

The Riemann Hypothesis (Riemann, 1859):

Every zero of $\zeta(s)$ in the strip $\operatorname{Re} s \in [0, 1]$ has real part $\frac{1}{2}$.

Implications for Prime-Counting Estimates

Riemann sketched the details of an argument that showed how the zeros of $\zeta(s)$ influenced estimates for $\pi(x)$. This sketch was made rigorous[3] by von Mangoldt in 1895, who proved that

$$\psi(x) = \frac{1}{2\pi i}\int_{(\sigma)} \left(-\frac{\zeta'(s)}{\zeta(s)}\right)\frac{x^s}{s}ds = x - \sum_{\rho}\frac{x^\rho}{\rho} + O(1),$$

in which ρ runs through the zeros of $\zeta(s)$. Note that:

1. We have $\psi(x) \sim x$ (equivalent to the PNT) if and only if $\zeta(s) \neq 0$ on the line $\mathrm{Re}s = 1$.
2. The Riemann Hypothesis gives $\psi(x) = x + O(x^{\frac{1}{2}+\epsilon})$ for all $\epsilon > 0$.

[3] Convergence issues, mostly.

A Proof of the Prime Number Theorem

A full proof of the Prime Number Theorem came soon thereafter by Hadamard and de la Vallée-Poussin (independently in 1896).

In their proofs, they show that $\zeta(s) \neq 0$ on the line $\mathrm{Re} s = 1$ and further justify convergence in Riemann's explicit formula.[4]

The Riemann Hypothesis, often described as one of the most important problems in mathematics, appeared on Hilbert's list of 23 problems in 1900 and as one of the seven Millenium Problems. It remains open.

[4] The sum is only conditionally convergent. Later-generational proofs would simplify the sum via regularization.

Questions?