Kepler's Conjecture

Problems from the History of Mathematics

Lecture 20 — April 9, 2018

Brown University

Problem History

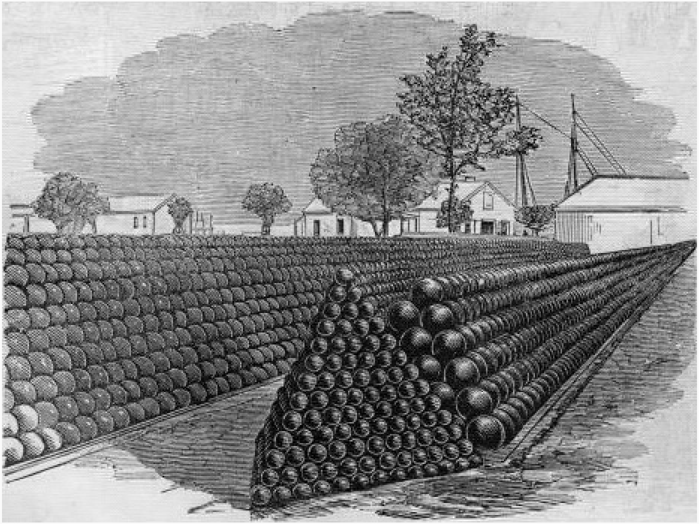
The story of Kepler's Conjecture begins with the English mathematician and polymath Thomas Harriot (c. 1560-1621) and his tenure as scientific advisor to Sir Walter Raleigh on a 1585 trip to the colony of Roanoke.

Sometime during this trip, Raleigh asked Harriot about the optimal stacking configuration of cannonballs on the deck of a ship.[1]

Harriot concludes that the optimal packing is the one you might expect.

[1] This anecdote perfectly describes the undertones of early trans-Atlantic expeditions.

A Close-Packing of Cannonballs



A depiction of stacked cannonballs at Fortress Monroe in 1861.

Problem History and Statement

Harriot's continued interest in sphere-packing is visible in his later work and his communication with Johannes Kepler.

In 1611, Kepler published *On the Six-Cornered Snowflake*, which included a conjecture for optimal sphere packings.

Conjecture (Kepler, 1611):

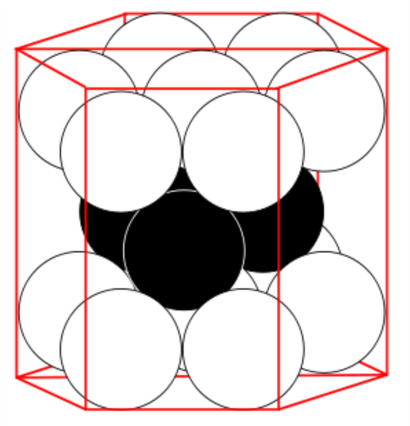
The optimal density of sphere packings is obtained by a packing called the cubic close packing. This density is $\pi/\sqrt{18} \approx 0.74048$.

This packing is also known as the face centered cubic (FCC) packing because sphere centers are the 8 vertices and 6 face centers of a cube.

Uniqueness?

Removing a finite number of spheres from the FCC packing gives a new configuration with the same density. There are many optimal packings.

Of these, a second is particularly interesting: the hexagonal close packing (HCP), the only other optimal lattice-based packing.



Comparison to Results in Lower Dimension

Circle Packing

Sphere packing is not particularly interesting in dimension one.

In dimension two, the optimal packing is the regular hexagonal packing[2], which has density $\pi/\sqrt{12} \approx 0.9069$.

This result is known as Thue's Theorem (1890), but is sometimes attributed to Lagrange (1773). Either way, it has an elementary proof.

The proof that the regular hexagonal packing is uniquely optimal *among lattice-based packings* is particularly easy.

[2]This packing appears in the 2-dimensional cross-sections of the FCC/HCP packings.

The Kissing Problem

One of the reasons that it's easy to move from lattice-based packings to general packings in dimension two is that it's very easy to bound the maximal number of unit circles tangent to a common unit circle.

In dimension three, this question is a lot harder.

Conjecture (Newton):

The maximal number of non-overlapping unit spheres tangent to a common unit sphere is 12.

This conjecture stems from a disagreement between Newton and Gregory (who thought that 13 might be possible). It is trivial to show that the kissing number in dimension 3 is at most 14.

Newton was vindicated in 1953 by Schütte and Van der Waerden.[3]

[3] Although many incomplete proofs, some due to Newton, predated this.

A Proof of Kepler’s Conjecture

Voronoi Cells and Delauney Triangulations

Before diving into the proof of Kepler's Cojecture, we introduce two fundamental concepts from discrete geometry. Let Λ be a discrete collection of points (in $\mathbb{R}^n$, say).[4]

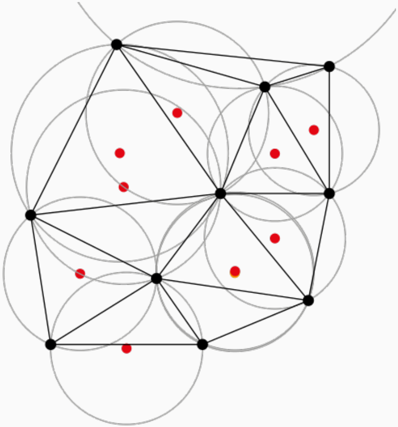
The Voronoi cell of $v \in \Lambda$ is the locus of points in $\mathbb{R}^n$ which lie closer to v than to any other point in Λ.

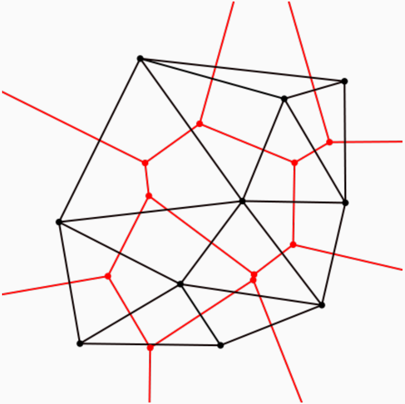
A Delauney triangulation is a triangulation of Λ in which no point $v \in \Lambda$ lies **strictly inside** one of the triangles of the triangulation.

[4] In our applications, Λ corresponds to the set of centers of the spheres in our packing.

Voronoi Cells and Delauney Triangulations

These two ideas are related: the Delauney triangulation represents a dual graph of the Voronoi diagram.





The Contributions of Tóth

In 1953, Lászlo Fejes Tóth considered the Voronoi cells associated to optimal sphere packings. Tóth's Dodecahedral Conjecture asserts that the volume of the associated Voronoi cells is bounded below by the volume of the regular dodecahedron of inradius 1.

This would imply that the optimal density of sphere packing is

$$\leq \frac{\text{volume of the inscribed sphere}}{\text{volume of the dodecahedron}} \approx 0.754697$$

Tóth showed that the Dodecahedral Conjecture would follow from explicit bounds on the proximity of a thirteenth sphere in the Kissing Problem.

The Contributions of Tóth

Tóth's Conjecture corresponds to extracting information from a single Voronoi cell. By considering many (ie. at most 13) Voronoi cells all at once, more progress could be made.

Tóth proved that the Kepler Conjecture would follow from the claim that a particular weighted average of volumes of Voronoi cells always exceeded the volume of the rhombic dodecahedron. Notably, Tóth's reduction was the first step which suggested that the Kepler Conjecture could be viewed as an optimization problem in **finitely many** variables.

To justify the reduction to at most 13 Voronoi cells, Tóth needed an effective bound for the sum of the distances from a central sphere to its thirteen nearest neighbors.

Hales' Proof of the Kepler Conjecture

Tóth's other contribution to the Kepler Conjecture was the suggestion that computers might assist in solving these optimization problems.

In 1922, Thomas Hales (and his graduate student Samuel Ferguson) began a research program to do exactly this.

Hales considers the space DS of decomposition stars, the unions of Delauney triangles which share a common vertex, and proves that DS embeds into a finite-dimensional space. The Kepler Conjecture is reduced to the claim that

$$\sqrt{32} \leq \mathrm{vol}(\Omega(D))$$

for all $D \in DS$, in which $\Omega(D)$ is the Voronoi cell attached to the decomposition star D. (Here, $\sqrt{32}$ is the volume of the fundamental region determined by either of the FCC or HCP lattices.)

Hales' Proof of the Kepler Conjecture

Roughly, Hales and Ferguson optimized by minimizing a function of 150 variables using computer-assisted linear programming. The proof required analysis of over 5000 cases and took six years.

The paper was submitted to *Annals of Mathematics*, where it passed to a board of 12 referees. After 4 years of review, the proof was accepted.

The proof was reported as correct with '99% certainty' by the committee. Perhaps unsatisfied with this, Hales set out in 2003 to produce a formal proof of the Kepler Conjecture.

This program, called FlysPecK, was estimated by Hales to take 20 years, but finished in 2014. The formal proof was accepted to *Forum of Mathematics, Pi* in 2017.

Questions?