BROWN

The Seven Bridges of Königsberg

Problems from the History of Mathematics

Lecture 18 — April 2, 2018

Brown University

The Seven Bridges of Königsberg

The city of Königsberg, Prussia[1] straddles the Pregel River.[2] As the story goes, it became a popular pastime in early 18th century Königsberg to spend Sunday walking through the city and visiting each of its seven bridges.

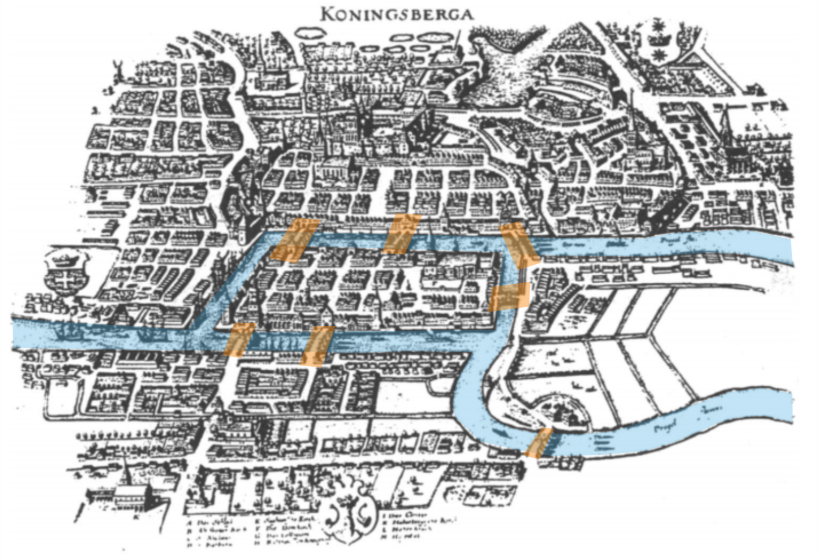
This led to a recreational problem which became locally famous:

The Seven Bridges of Königsberg

Is it possible to tour the city of Königsberg by crossing each of the seven bridges in the city exactly once?

[1] Now, Kaliningrad, Russia
[2] Now, Progolya River

The Seven Bridges of Königsberg



Enter Euler

The citizens of Königsberg suspected that such a tour was impossible but were unable to prove this. Fortunately, for them, Prussia was close to St. Petersburg, the residence of Euler since 1729.

It's not clear how Euler learned of the Bridges Problem. At some point, though, Euler discussed the problem with Carl Leonhard Gottlieb Ehler, the mayor of Danzig (now Gdansk). In 1736, Ehler wrote:

> *You would render to me and our friend Kühn[3] a most valuable service, putting us greatly in your debt, most learned Sir, if you would send us the solution, which you know well, to the problem of the seven Königsberg bridges, together with a proof. It would prove to be an outstanding example of the calculus of position, worthy of your great genius.*

[3] Heinrich Kühn, a local mathematics professor

The Calculus/Geometry of Position

This reference to the calculus of position appears to date back to a letter from Liebniz to Huygens:

> *I am not content with algebra, in that it yields neither the shortest proofs nor the most beautiful constructions of geometry. Consequently, in view of this, I consider that we need yet another kind of analysis, geometric or linear, which deals directly with position, as algebra deals with magnitudes...*
>
> — *Liebniz, 1679*

Euler's opinion on whether the Bridges of Königsberg problem fits into Liebniz's calculus of position is murky; he waffles on the topic in letters from 1736.

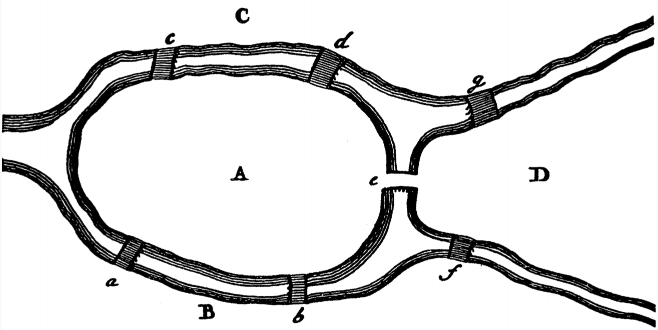
Today, we'd interpret the calculus of position as a form of topology.

Euler’s Solution

Euler's Solution

Euler solves the Bridges of Königsberg through a series of simplifications and observations.

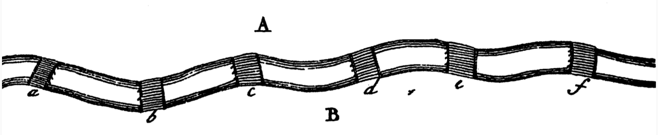
1. It's enough to consider only the land masses and the bridges. Euler includes the following, abstracted diagram[4] in his 1736 paper:



[4] In other words, its enough to study the problem topologically.

Euler's Solution

2. Euler notates paths by concatenating the names for land masses. Referring back to the diagram from before, we need an 8-letter string with two adjacent A/B pairs, two adjacent A/C pairs, and one each of C/D, A/D, and B/D pairs.
3. Euler shows that a land mass connected via k (odd) bridges must be visited $(k+1)/2$ times in a tour. He shows this by simpler example:



4. Thus A appears three times, B twice, C twice, and D twice. But this path corresponds to a 9-letter string, a contradiction!

Euler's Generalization

The Königsberg bridge configuration is special because each land mass is connected to the rest with an odd number of bridges. In the same paper, Euler also addressed the general case, in which a collection of n land masses are joined by any number of bridges.

Theorem (Euler, 1736):

1. If more than two land masses are joined to the rest by an odd number of bridges, then such a journey is impossible.
2. If exactly two land land masses are accessible by an odd number of bridges, the journey is possible if it starts in one of these two areas.
3. If each area is accessible via an even number of bridges, the journey can be accomplished starting anywhere.

Modern Terminology

The Bridges of Königsberg problem[5] is famous for creating the field now known as graph theory.[6] In modern terminology, we construct a graph in which

1. Land masses are represented by vertices/nodes
2. Bridges are represented by edges

The number of edges emanating from a vertex is called the degree of the vertex. Euler's theorem relates the number of vertices in a graph of odd degree with the existence of a complete, non-intersecting path now known as an Eulerian path in his honor.

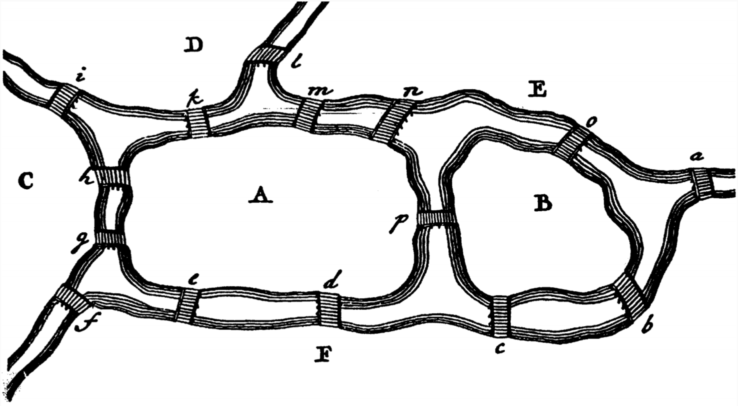
---

[5] Together with the Knight's Tour Problem.

[6] Although graph theory is typically thought of as a subfield of combinatorics as opposed to a subfield of topology.

Solvable Configurations

For contrast, Euler gives an example of a configuration with a solution:



(One solution path crosses the bridges $a, \ldots, l$ in alphabetical order.)

An Algorithm

One algorithm for the construction of Eulerian paths dates to 1883 and is known as Fleury's Algorithm. To construct an Eulerian path,

1. Begin at any odd degree vertex, if one exists. Otherwise start at any vertex.
2. To choose the next edge, prioritize edges whose deletions preserve graph connectivity. This is possible unless the vertex has only one undeleted edge remaining. In this case, take that edge. Either way, traverse it then delete it. Repeat.

Edges whose deletions disconnect the graph are known, confusingly, as bridges. Thus Fleury's algorithm is the algorithm which *burns edges but tries to not burn bridges.*

Faster (but less elegant) algorithms exist, including Hierholzer's algorithm from 1873.

Questions?