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# Integration of Elementary Functions

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Problems from the History of Mathematics

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# Early Successes in Integration

In the time between Archimedes' *Method* (which found the area under a parabola) and the dawn of calculus, several important results about integration were proven.

Many of these advances came from summing the series

$$1^k + 2^k + \dots + n^k.$$

**Ex.** In the case  $k = 2$ , we have  $1^2 + 2^2 + \dots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ . It follows that the upper Riemann sum approximation to the area under  $y = x^2$  over  $[0, 1]$  using rectangles of width  $1/n$  is

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k^2}{n^3} = \lim_{n \rightarrow \infty} \frac{\frac{1}{6}n(n+1)(2n+1)}{n^3} = \frac{1}{3}.$$

The lower estimate agrees, so the integral exists and equals  $\frac{1}{3}$ .

## Early Successes in Integration

The Arab mathematician al-Haytham (c. 965-1039) computed the sum  $1^k + \dots + n^k$  for  $k = 1, 2, 3, 4$  and used these results to compute the volumes of certain solids of revolution.<sup>1</sup>

Cavalieri extended these results to  $k \leq 9$  in 1635 and conjectured that

$$\int_0^a x^k dx = \frac{a^{k+1}}{k+1}$$

for all positive integers  $k$ . (This was proven by Fermat, Descartes, and Roberval around 1630.) By considering  $y^n = x^m$ , Fermat extended this to positive fractional powers. It was conjectured to hold for all powers (except for  $k = -1$ ).

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<sup>1</sup>One such surface, the rotation of  $y = 1/x$ , is called Gabriel's Horn (also Torricelli's Trumpet). It has finite volume but infinite surface area and was considered paradoxical in its time.

# Quadrature of the Hyperbola

The case  $k = -1$ , corresponding to area under the hyperbola  $y = 1/x$ , would not be resolved until 1649. In this year, Grégoire de Saint-Vincent and Alphonse Antonio de Sarasa derived (in modern terms) that

$$\int_a^b \frac{dt}{t} = \int_{ca}^{cb} \frac{dt}{t}.$$

Thus the definite integral of  $1/t$  from  $x = 1$  to  $x = u$  is a function which satisfies  $A(uv) = A(u) + A(v)$ . This is the same functional equation satisfied by the logarithm and  $A(x)$  is the first appearance of the **natural logarithm**.

**Note:** Riemann sums with rectangles of uniform width give results like

$$\int_1^2 \frac{dt}{t} = \lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{1}{n+k}$$

and do not aid in finding closed forms. A more useful division uses rectangles over the ranges  $[\lambda^{k/n}, \lambda^{(k+1)/n}]$ , with  $\lambda > 1$  small.

# The Calculus of Newton

Newton and Leibniz are both credited with the invention of calculus, but the nature of their work differs greatly. **Newton's** calculus builds on the idea of power series. Once series expansions are known,<sup>2</sup> integration and differentiation reduce to integration and differentiation for polynomials.

*Since the operations of computing in numbers and with variables are closely similar... I am amazed that it has occurred to no one (if you except Mercator with his quadrature of the hyperbola) to fit the doctrine recently established for decimal numbers in similar fashion to variables, especially since the way is then open to more striking consequences... Its operations of Addition, Subtraction, Multiplication, Division and Root extraction may be easily learnt from [arithmetic].* — Newton (1671)

The result that Newton alludes to is

$$\int_0^x \frac{dt}{1+t} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

first published by Mercator in 1668.

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<sup>2</sup>For example, by manipulating Newton's binomial series.

# The Calculus of Leibniz

Conversely, **Leibniz**'s calculus was one of infinitesimals. In a practical sense, though, Leibniz strove to recognize the integrals and derivatives of known functions in terms of other known functions.<sup>3</sup>

It is only in Leibniz's paradigm that today's problem makes sense:

## Question:

Is it possible to find **closed form solutions** for the anti-derivatives of **elementary functions**?

The definitions of both 'closed form' and 'elementary' have changed over time. For now, we'll take each to mean functions obtained by combining powers, exponentials, trigonometric functions, and their inverse functions, with rational functions.

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<sup>3</sup>E.g. Leibniz was interested in rules for derivatives of sums, products, and quotients.

# Antiderivatives of Rational Functions

Integrals of polynomials were understood as early as 1630. The next set to be systematically understood were the rational functions.

It was known early on that polynomial long division and partial fraction decomposition could be used to simplify the general case. Yet the theory was still a mess because the fundamental theorem of algebra was not known. For example, Leibniz (1701) could not handle

$$\int \frac{dz}{z^4 + 1}$$

because he believed, erroneously, that  $z^4 + 1$  was irreducible over  $\mathbb{R}$ .

By assuming the FTOA, Bernoulli was able to give a complete treatment of rational functions in 1703. (Ostrogradsky credits Bernoulli for the first integration algorithm.)

# Antiderivatives of Algebraic Functions

The simplest non-rational algebraic functions are conics. These are solved by trig substitution, so the first interesting cases are integrals of the form

$$\int \frac{dx}{\sqrt{f(x)}},$$

in which  $f(x)$  is a cubic.<sup>4</sup> These represent the areas under **elliptic curves**.

Euler proved transformation laws for integrals of this form which would not be fully understood until the work of Gauss (1800) and Jacobi (1834) relating them to group laws on elliptic curves.

Strangely, this connection was anticipated by Leibniz as early as 1702:

*I... remember having suggested (what could seem strange to some) that the progress in our integral calculus depended in good part upon the development of that type of arithmetic which, so far as we know, Diophantus has been the first treat systematically.* — Leibniz

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<sup>4</sup>Or quartic. The cases are essentially the same.



# Elliptic Integrals

Nowadays, we define an elliptic integral as any integral of the form

$$\int R(t, \sqrt{P(t)}) dt,$$

in which  $R(x, y)$  is a rational function and  $P(t)$  is either a cubic or quartic polynomial. Elliptic integrals are so-called because they include the integral which determines the arc length of the ellipse  $y = \sqrt{1 - t^2}/k$ :

$$\int \frac{\sqrt{1 + (k^2 - 1)t^2}}{\sqrt{1 - t^2}} dt.$$

(This is related to the **elliptic integral of the second kind**.)

A simpler elliptic integral to investigate is

$$u(x) = \int_0^x \frac{dt}{\sqrt{1 - t^4}}.$$

# Elliptic Functions

The theory of elliptic integrals only took off after Gauss realized that one should not study  $u(x)$  but rather the inverse function  $x$  as a function of  $u$ .

For example,

$$u(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}}.$$

should be compared to to the integral of  $1/\sqrt{1-t^2}$ ; namely, to arcsine. Just as we study arcsine by studying sine, we should study  $u(x)$  through its inverse function.

And just like sine,  $\operatorname{arcu}(x)$  is periodic. In fact,  $\operatorname{arcu}(x)$  is **doubly periodic** – it has a second period which is complex. Thus elliptic integrals led to the theory of **elliptic functions**, doubly periodic functions on the complex plane, in the nineteenth century.

# Liouville's Theorem of Differential Algebra

The first tool that could be used to prove that certain functions *did not* have elementary antiderivatives was proven by Liouville c. 1833-1841.<sup>5</sup>

We define a **differential field** as a field (of functions) which is closed under differentiation. A field extension  $G/F$  of differential fields is called

- a. **logarithmic** if  $G = F(t)$  for some transcendental function  $t$  such that  $t' = s'/s$  for some  $s \in F$ .
- b. **exponential** if  $G = F(t)$  for some transcendental function  $t$  such that  $t' = ts'$  for some  $s \in F$ .

An extension is called an **elementary differential extension** if  $G/F$  is a tower of algebraic, logarithmic, and exponential extensions.

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<sup>5</sup>Some sources give 1838.

# Liouville's Theorem of Differential Algebra

## Theorem (Liouville, 1833-1841):

Suppose that  $G/F$  is an elementary differential extension. If  $G$  contains an antiderivative of  $s$ , then there exist  $u_j, v \in F$  and  $c_j \in \mathbb{C}$  such that

$$s = v' + \sum c_j \frac{u_j'}{u_j}.$$

With Liouville's 'structure theorem' it is possible to prove that certain functions, like  $e^{-x^2}$  and  $1/\log x$  have no elementary antiderivative.

## Corollary:

Choose  $f, g \in \mathbb{C}(x)$  with  $f \neq 0$  and  $g$  non-constant. Then  $f(x)e^{g(x)}$  can be integrated in elementary terms if and only if there exists a rational function  $R \in \mathbb{C}(x)$  such that  $R'(x) + g'(x)R(x) = f(x)$ .

# The Risch Algorithm

The question of the existence of elementary antiderivatives was settled conclusively in 1968 by the work of Robert Henry Risch.

The result, the **Risch Algorithm**, is a method to

1. decide if a function  $f(x)$  has an elementary antiderivative; and
2. if  $f(x)$  does, determine its indefinite integral

The complete description of the Risch algorithm occupies over 100 pages. For this reason, it was notoriously hard to implement in computer algebra packages. The first implementation is probably due to James Davenport<sup>6</sup> sometime between 1979 and 1985.

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<sup>6</sup>Son of number theorist Harold Davenport

**Questions?**