Approximating Pi

Problems from the History of Mathematics

Lecture 15 — March 14, 2018

Brown University

Introduction

The oldest approximations to π appear in the preserved works of the ancient Babylonians and Egyptians. One example problem (from the Rhind Papyrus) follows:

Problem (RMP 50):

What is the area of a field with a diameter of 9 khet?

Solution: Take $\frac{1}{9}$ from the diameter, leaving 8. Multiply this number by itself. The answer is 64 setjat (square khet). $\square$

We conclude that

$$64 \approx \pi(\tfrac{9}{2})^2 \quad \Longrightarrow \quad \pi \approx \tfrac{256}{81} \approx 3.16049.$$

Caution: It's easy to look at approximations like these and assume too much of the state of mathematics at the time. At this point in history there had been no attempt to codify a single value of π for use in future problems. Associating these cultures with their best guesses is misleading.

The Era of Polygonal Approximations

Archimedes and the Method of Exhaustion

Earlier in the course, we discussed Archimedes approximation

$$3.14085 \approx 3 + \tfrac{10}{71} < \pi < 3 + \tfrac{1}{7} \approx 3.14286$$

using the perimeters of inscribed and circumscribed 96-gons.

Archimedes' approach (and a related one using areas instead of perimeters) is significant for two reasons:

1. It is effective: we obtain upper and lower bounds and can estimate the error in our approximation.
2. It is algorithmic: it can be used to prove estimates of any precision.

Liu Hui's Formula

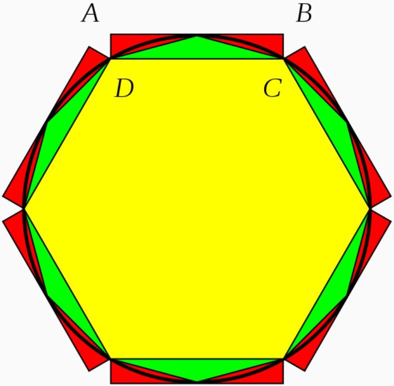
Polygonal methods continued to be used until the seventeenth century. In this time, the greatest advance was due to Liu Hui (c. 225 - c. 295).

Let A_N denote the area of the inscribed N-gon. As the figure shows,

$$A_{2N} < \pi < A_N + 2(A_{2N} - A_N).$$

It remained to relate the areas of the N-gon and the $2N$-gon. Let s_N denote the side length of the inscribed N-gon. Liu Hui showed that

$$s_{2N} = \sqrt{2 - \sqrt{4 - s_N^2}}.$$



Polygonal Approximation

Liu Hui used his method on the 48- and 96-gons to prove that

$$3.141024 < \pi < 3.142704.$$

Liu Hui and other authors then improved the polygonal method[1] and pushed the computational limits of the day:

N	Digits	Author	Year
$2^{10} \cdot 3$	4	Liu Hui	c. 265
$2^{12} \cdot 3$	7	Zu Chongzhi	c. 480
$2^{28} \cdot 3$	16	Jamshīd al-Kāshī	1424
2^{62}	35	Ludolph van Ceulen	c. 1610
10^{40}	38	Christoph Grienberger	1640

[1] For example, by improving the bounds $A_{2N} < \pi < A_N + 2(A_{2N} - A_N)$.

Formulas of Machin Type

The Gregory–Liebniz Series

The Gregory–Liebniz series is the infinite series

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots,$$

which was first discovered in India between 1400 and 1500 as a specialization of the Taylor series for arctangent.[2]

This series is not actually useful for computing π, but variants using $\arctan(1/\sqrt{3})$ gave linear convergence.[3] Abraham Sharp used the $z = 1/\sqrt{3}$ series to approximate π to 71 digits in 1699.

Error terms can be estimated easily because these are alternating series.

[2] And then rediscovered by Gregory in 1671 and Leibniz in 1674.
[3] This means that $O(n)$ digits can be obtained from $O(n)$ terms.

Machin's Formula

In 1706, John Machin applied the arctangent addition formula

$$\arctan \frac{a_1}{b_1} + \arctan \frac{a_2}{b_2} = \arctan \frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2}$$

to rapidly accelerate implementations of the Gregory–Liebniz series. Machin proved that

$$\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}.$$

The rate of convergence is exponential in $1/5$ and gives a constant speed-up over series using $1/\sqrt{3}$ while avoiding square-roots.

William Shanks published 707 digits (of which the first 527 were correct) in 1873 using Machin's formula. Daniel Ferguson obtained 620 correct digits in 1946 using Machin's formula and a desk calculator.

Machin-like Formulas

Machin's original formula is not unique. Other formulas obtained from the arctangent addition law include

a. $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$ (Euler)
b. $\frac{\pi}{4} = 2 \arctan \frac{1}{2} - \arctan \frac{1}{7}$ (Hermann)
c. $\frac{\pi}{4} = 2 \arctan \frac{1}{3} + \arctan \frac{1}{7}$ (Vega)

Machin-like formulas with more than two terms are used to approximate π to this day. For example, the 2002 record of 1.2411 trillion digits employed the Machin pair

$$\frac{\pi}{4} = 12 \arctan \frac{1}{49} + 32 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239} + 12 \arctan \frac{1}{110443}$$
$$\frac{\pi}{4} = 44 \arctan \frac{1}{57} + 7 \arctan \frac{1}{239} - 12 \arctan \frac{1}{682} + 24 \arctan \frac{1}{12943}$$

Other Algorithms to Compute Pi

Ramanujan–Sato Series

In the last few decades, the series used to compute π have been a mix of Machin-like formulas and Ramanujan–Sato series. These latter series stem from the work of Ramanujan in modular forms and are known for their unexpected appearance; e.g.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}.$$

The Chudnovsky brothers developed a Ramanujan–Sato formula in 1987 which was used to break digit records in 1989, 2009, and 2011.

Following the release of the program y-cruncher in 2010, all records have been set using Ramanujan–Sato series. The current record is 22.4 trillion digits, set in November 2016 by Peter Trueb.

Iterative Methods

A second algorithm used in recent years to compute π is called the Gauss–Legendre algorithm. The algorithm is based on the algorithm for computing the arithmetic-geometric mean and runs as follows:

The Gauss–Legendre Algorithm:

0. Let $a_0 = 1$, $b_0 = \frac{1}{\sqrt{2}}$, and $t_0 = \frac{1}{4}$.
1. Apply the relations

$$a_{n+1} = \frac{a_n + b_n}{2}, \quad b_{n+1} = \sqrt{a_n b_n}, \quad t_{n+1} = t_n - 2^n (a_n - a_{n+1})^2.$$

2. Then $\pi \approx \dfrac{(a_{n+1} + b_{n+1})^2}{4t_{n+1}}$.

The number of correct digits produced by this algorithm roughly doubles every iteration. This speed comes at the cost of extreme memory usage, however.

Questions?