Logarithms

Problems from the History of Mathematics

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Brown University

Introduction

Before we jump right into the history of the logarithm, I should probably dispel some skepticism. You may be wondering: How are logarithms a problem from the history of mathematics?

The answer is that logarithms were in fact a **solution** to many problems plaguing science in the late sixteenth century. The rapid development of astronomy, long-distance navigation, cartography, and surveying lead to enormous amounts of data, and this data needed processing.

"[the logarithm has] doubled the life of the astronomer." – Laplace[1]

Over the next two centuries, the logarithm moved from a computational tool to an analytic one. It thus plays a role in the development of calculus and inverse function theory.

[1] Probably.

Early Appearance of Logarithms

At its core, the logarithm is a tool for relating addition and multiplication. Prior to this, though, we see the logarithm as a way to relate geometric progressions to arithmetic progressions.

1. Euclid discusses both arithmetic and geometric progressions (eg. in formulas for perfect numbers).
2. Archimedes writes the *Sand Reckoner*, which prefigures the complete decimal system we call the Hindu-Arabic system.
3. Virasena (816) computes the number of times you can halve an integer before reaching an odd number. (Computes the 2-exponent.) This was also done by Stifel (1544), who coined the term exponent.

Multiplication with Tables

The practice of expediting multiplication by referencing tables also predates the introduction of the logarithm. One method is called quarter square multiplication, which dates from Bablyon (c. 2000-1600 BC).

This applies the identity

$$(x+y)^2 - (x-y)^2 = x^2 + 2xy + y^2 - x^2 + 2xy - y^2 = 4xy$$

to reduce multiplication of xy to two squaring operations. The squares could then be read from a table to speed up computation.

Note: It's unreasonable to write a table that lists products xy because the number of possiblities grows quadratically in the number of inputs.

The quarter squares algorithm reduces the binary operation $\times$ to the binary operation $+$ and the single-variable function $x \mapsto x^2$.

Prosthaphaeresis

In the sixteenth century, long-distance navigation at sea relied on tables of measurements of the positions of stars at various times. Navigators used spherical trigonometry to compute positions, which often involved lengthy multiplications and trigonometric tables.

An enormous chunk of an astronomer's time was spent long multiplying.

The first method used to expedite this work was called prosthaphaeresis[2], which arose circa 1580 and was popularized by Tycho Brahe. We recognize this technique in terms of the trigonometric angle addition/ subtraction formulas:

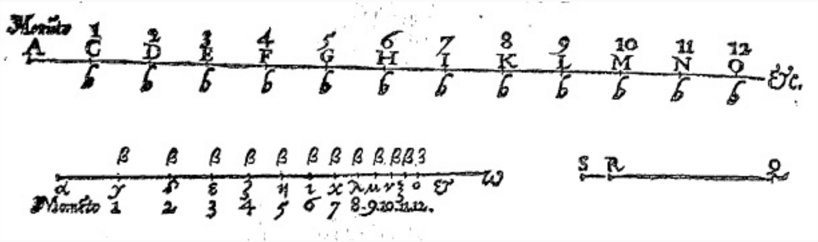
$$\sin\alpha\sin\beta = \frac{1}{2}\left(\cos(\alpha-\beta) - \cos(\alpha+\beta)\right), \quad \text{etc.}$$

Note: In spherical trigonometry, arc lengths appear as products of trigonometric functions.

[2] *addition and subtraction*

Napier's Logarithms

Napier first published his work on logarithms in 1614, under the title *A Description of the Wonderful Table of Logarithms*. Napier was aware of prosthaphaeresis but developed a new system for logarithms in a kinematic framework.



Napier imagines two particles, one moving on an infinite line at uniform speed and another on a finite length segment with velocity proportional to the distance to the terminal point.

Napier's Tables

Napier generated numerical data for his tables by sampling the motion of these particles in small discrete steps. In practice, Napier computed

$$(1 - 10^{-7})^L$$

for $L \in [1, 100]$, for $L/100 \in [1, 50]$, and for various small L divisible by 1000 and other powers of ten. By combining these computations,[3] Napier found $(1 - 10^{-7})^L$ for millions of values of L.

Inverting these tables, one can find the 'artificial number' L which gave

$$N = 10^7(1 - 10^{-7})^L,$$

for any N between 5×10^6 and 10^7. In modern notation,

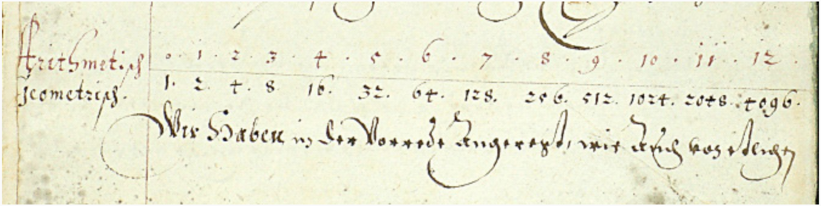
$$L = \log_{(1-10^{-7})}\left(\frac{N}{10^7}\right).$$

[3] Over the course of twenty years.

Bürgi’s Logarithms

Around the same time, a Swiss clockmaker named Joost Bürgi pursued the problem of multiplication and the problem of having many tables for various functions.

Unlike Napier, Bürgi grounds his work in the relation between arithmetic and geometric progressions. Early on, this is illustrated with powers of 2:



Noting that powers of two grew too quickly, Bürgi’s actual tables[4] were produced with the common ratio 1.0001.

[4] Totaling 23030 entries over 58 pages.

Logarithms as Areas

In 1649, Grégoire de Saint-Vincent and his student Alphonse Antonio de Sarasa pointed out that the areas under the hyperbola $xy = 1$ over the intervals $[a, b]$ and $[c, d]$ were equal when $a/b = c/d$.

In particular, if $A(t)$ denotes the area under the curve from $x = 1$ to $x = t$, then $A(uv) = A(u) + A(v)$. This gave a geometric meaning to the logarithm which helped ground it outside of computation.

More significantly, this work marks the first appearance of the natural logarithm, a term which would first appear in Nicholas Mercator's *Logarithmo-technia* from 1668.

The logarithm would be used by Euler to define the exponential function around 1730:

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \qquad \log(x) = \lim_{n\to\infty} n(x^{1/n} - 1).$$

Work by Euler in 1748 identifies them as inverse functions.

Questions?