The Classification of Pythagorean Triples

Problems from the History of Mathematics

Lecture 1 — January 24, 2018

Brown University

The Pythagorean Theorem

The Pythagorean Theorem in Egypt

While the first proof of the Pythagorean Theorem comes from ancient Greece, special cases of this theorem were known in Egypt and Babylon.

However, the purported role of the Pythagorean Theorem in Egyptian architecture appears to be overblown. Cooke[1] writes:

> *The Pythagorean theorem.* Inevitably in the discussion of ancient cultures, the question of the role played by the Pythagorean theorem is of interest. Did the ancient Egyptians know this theorem? It has been reported in numerous textbooks, popular articles, and educational videos that the Egyptians laid out right angles by stretching a rope with 12 equal intervals knotted on it so as to form a 3–4–5 right triangle. What is the evidence for this assertion? First, the Egyptians *did* lay out very accurate right angles. Also, as mentioned above, it is known that their surveyors used ropes as measuring instruments and were referred to as *rope-fixers* (see Plate 7). That is the evidence that was cited by the person who originally made the conjecture, the historian Moritz Cantor (1829–1920) in the first volume of his history of mathematics, published in 1882.

---

[1] *A Brief History of Mathematics*, p. 237.

Still further, Berlin Papyrus 6619 contains a problem in which one square equals the sum of two others. It is hard to imagine anyone being interested in such conditions without knowing the Pythagorean theorem. Against the conjecture, we could note that the earliest Egyptian text that mentions a right triangle and finds the length of all its sides using the Pythagorean theorem dates from about 300 BCE, and by that time the presence of Greek mathematics in Alexandria was already established. None of the older papyri mention or use by implication the Pythagorean theorem.

On balance, one would guess that the Egyptians *did* know the Pythagorean theorem. However, there is no evidence that they used it to construct right angles, as Cantor conjectured. There are much simpler ways of doing that (even involving the stretching of ropes), which the Egyptians must have known. Given that the evidence for this conjecture is so meager, why is it so often reported as fact? Simply because it has been repeated frequently since it was originally made. We know precisely the source of the conjecture, but that knowledge does not seem to reach the many people who report it as fact.

The Pythagorean Theorem in Egypt

The Berlin Papyrus 6619 that Cooke refers to consists of papyrus fragments found in Saqqara dating to c. 1800 BC. These fragments contain the following problem:

Berlin Papyrus 6619, Problem 1:

The area of a square of 100 is equal to that of two smaller squares. The side of one is $1/2 + 1/4$ the side of the other. What are the sides of the two unknown squares?

As Cooke suggests, it seems unlikely that the Egyptians would have been interested in this problem without knowledge of the Pythagorean theorem in some capacity.

Note: The expression $1/2 + 1/4$ is an example of an Egyptian fraction, which we will discuss in the next lecture.

The Pythagorean Theorem in Babylon

In contrast, the ancient Babylonians appear to have been aware of the Pythagorean theorem (in statement if not in proof) prior to the cultural influence of Greece in Babylon.

This theory is supported by a Babylonian tablet known as BM 85 196 (c. 1800 BC) which includes the following problem and its solution:

BM 85 196:

A beam of length $0; 30$ GAR is leaning against a wall. Its upper end is $0; 6$ GAR lower than it would be if it were perfectly upright. How far is its lower end from the wall?

Do the following: Square $0; 30$, obtaining $0; 15$. Subtracting $0; 6$ from $0; 30$ leaves $0; 24$. Square $0; 24$, obtaining $0; 9, 36$. Subtract $0; 9, 36$ from $0; 15$, leaving $0; 5, 24$. What is the square root of $0; 5, 24$? The lower end of the beam is $0; 18$ from the wall.

The Pythagorean Theorem in Greece

In Greece we see at last a formal proof of the Pythagorean Theorem.

The first[2] proof likely came from the Pythagorean school (c. 570-495 BC) and is in the classic style of dissection (ie. cut-and-paste) geometry.

It is best understood by illustration:

---
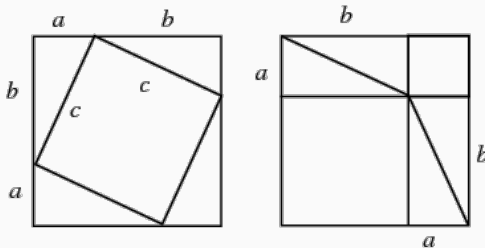
[2]In the least, it is the first proof introduced to western civilization. Independent proofs arose in China and India.

The Pythagorean Theorem in Greece

The Pythagorean Theorem appears with a new proof in Euclid's *Elements* as Proposition 47 of Book I. Since *Elements* does not introduce the idea of similarity until Books V-VI, a proof using shears is used instead.
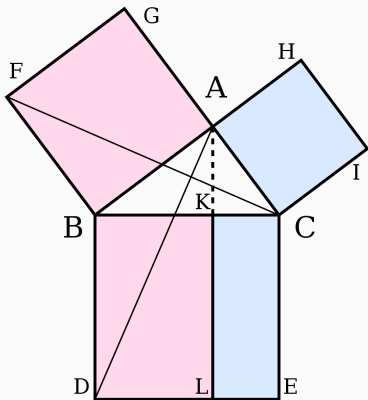
# Classification of Pythagorean Triples

Pythagorean Triples in Babylon

In ancient Egypt, we see that the general statement of the Pythagorean Theorem is predated by specific examples of Pythagorean triples, such as the $(3, 4, 5)$ triangle. The same appears to be the case in Babylon.

The primary artifact in support of this hypothesis is a clay tablet dated to about 1800 BC now known as Plimpton 322. Unearthed c. 1900 and sold by Edgar J. Banks to George Arthur Plimpton c. 1920, this tablet was bequeathed to Columbia University in 1930 where it remains today.
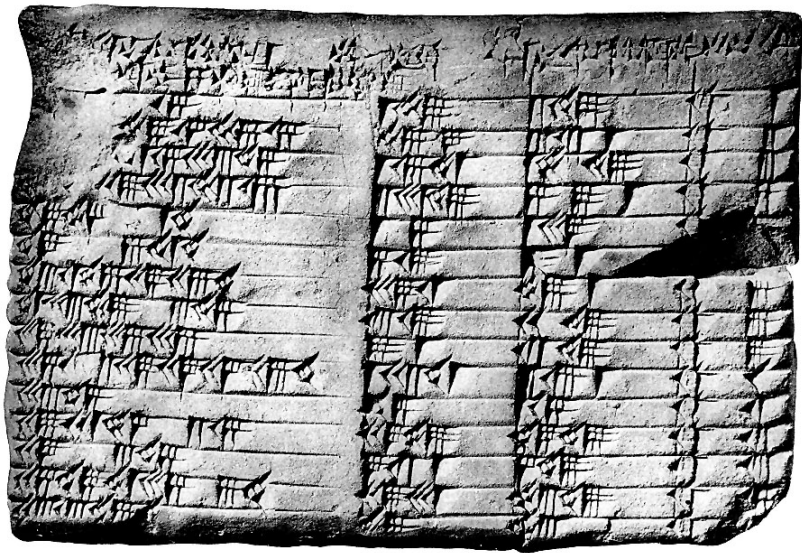
Plimpton 322

Correcting six probable scribal errors and translating into decimal notation, Plimpton 322 reads as follows:[3]

		Decimal Translation of Plimpton 322		
I. (damaged), d^2/l^2 or b^2/l^2	II. Square-side of the width, b	III. Square-side of the diagonal, d	IV. Its name	(Square-side of the length, l)
(1).9834028	119	169	1	120
(1).9491586	3367	4825	2	3456
(1).9188021	4601	6649	3	4800
(1).8862479	12,709	18,541	4	13,500
(1).8150077	65	97	5	72
(1).7851929	319	481	6	360
(1).7199837	2291	3541	7	2700
(1).6845877	799	1249	8	960
(1).6426694	481	769	9	600
(1).5861226	4961	8161	10	6480
(1).5625	45	75	11	60
(1).4894168	1679	2929	12	2400
(1).4500174	161	289	13	240
(1).4302388	1771	3229	14	2700
(1).3871605	28	53	15	45

---

[3] Eleanor Robson, *Neither Sherlock Holmes Nor Babylon*.

Plimpton 322

Math historian Otto Neugebauer posited a number-theoretic justification in his work[4] on Plimpton 322 in 1952. Neugebauer and Sachs suggested that the rows of Plimpton 322 were generated using the formula

$$(2mn, m^2 - n^2, m^2 + n^2),$$

but this theory has fallen out of favor since it does not explain the order of the rows of Plimpton and the choice of seed values m, n.

A more recent interpretation due to Robson is based on the Babylonians algorithmic treatment of quadratic equations via reciprocal pairs, pairs of integers which multiply to some power of 60. In this view, Plimpton 322 is more likely a teacher's aid than a work of research mathematics.
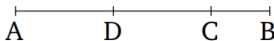
---

[4] *The Exact Sciences in Antiquity.*

Pythagorean Triples in Ancient Greece

The first general formula for producing integer Pythagorean triples is given in *Elements* as a Lemma before Proposition 29 in Book X.

To find two square numbers such that the sum of them is also square.

A   D   C   B

Let the two numbers AB and BC be laid down. And let them be either (both) even or (both) odd. And since, if an even (number) is subtracted from an even (number), or if an odd (number is subtracted) from an odd (number), then the remainder is even [Props. 9.24, 9.26], the remainder AC is thus even. Let AC have been cut in half at D. And let AB and BC also be either similar plane (numbers), or square (numbers)—which are themselves also similar plane (numbers). Thus, the (number created) from (multiplying) AB and BC, plus the square on CD, is equal to the square on BD [Prop. 2.6]. And the (number created) from (multiplying) AB and BC is square—inasmuch as it was shown that if two similar plane (numbers) make some (number) by multiplying one another then the (number so) created is square [Prop. 9.1]. Thus, two square numbers have been found—(namely,) the (number created) from (multiplying) AB and BC, and the (square) on CD—which, (when) added (together), make the square on BD.
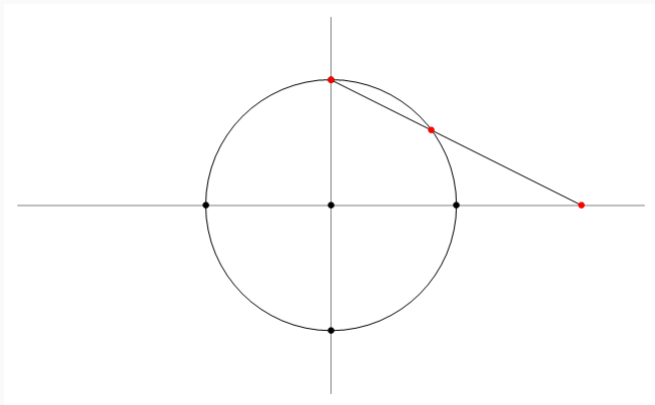
Pythagorean Triples in Ancient Greece

Geometrically, Euclid's proof visualizes $AB \cdot BC$ as a gnomon, the gap between two nested squares. This technique can be used to prove the Pythagorean Theorem and is especially helpful in this arithmetic setting because it reduces the task of finding three squares to finding just one.

It appears that Euclid's proof is actually a synthesis of two cases which appear in the work of the Pythagoreans and in Plato. In any case, the Pythagoreans would have been able to produce triples such as $(3, 4, 5)$ and $(5, 12, 13)$ by writing out a list of squares, computing the successive differences, and selecting only those whose difference was a square.

Classification via Stereographic Projection

A more recent proof of Euclid's formula uses the idea of stereographic projection, which provides a bijective mapping between the rational points on the circle and $\mathbb{Q} \cup \{\infty\}$.

Questions?