The Cattle Problem of Archimedes

Problems from the History of Mathematics

Lecture 7 — February 12, 2018

Brown University

Problem Statement

The Cattle Problem of Archimedes

The Cattle Problem refers to a Greek word problem written in a 44-line poem and rediscovered by Gotthold Lessing in a scroll within the Herzog August Library[1] in Wolfenbüttel, Germany in 1773.

The problem is attributed to Archimedes (c. 287 - 212 BC) and concerns the number of cattle in a herd owned by the Greek sun god Helios.

This number is defined only implicitly and the reader is challenged to find it as a show of mathematical skill.

[1] This is one of the oldest libraries to have never suffered a major loss to its collections.

The Cattle Problem , Pt. I

The problem itself is a bit long. It begins thus:

> If thou art diligent and wise, O stranger, compute the number of cattle of the Sun, who once upon a time grazed on the fields of the Thrinacian isle of Sicily, divided into four herds of different colours, one milk white, another a glossy black, a third yellow and the last dappled. In each herd were bulls, mighty in number according to these proportions: Understand, stranger, that the white bulls were equal to a half and a third of the black together with the whole of the yellow, while the black were equal to the fourth part of the dappled and a fifth, together with, once more, the whole of the yellow. Observe further that the remaining bulls, the dappled, were equal to a sixth part of the white and a seventh, together with all of the yellow.

The Cattle Problem, Pt. I

Let W denote the number of white bulls, B the number of black, D the number of dappled, and Y the number of yellow. Algebraically, we have

$$\begin{aligned} W &= (\tfrac{1}{2} + \tfrac{1}{3})B + Y = \tfrac{5}{6}B + Y \\ B &= (\tfrac{1}{4} + \tfrac{1}{5})D + Y = \tfrac{9}{20}D + Y \\ D &= (\tfrac{1}{6} + \tfrac{1}{7})W + Y = \tfrac{13}{42}W + Y \end{aligned}$$

Solving the linear system gives the parametrized family

$$W = 2226t, \quad Y = 891t, \quad D = 1580t, \quad B = 1602t,$$

with $t \in \mathbb{Z}$ (since we require integer solutions).

But we've only just begun:

These were the proportions of the cows: The white were precisely equal to the third part and a fourth of the whole herd of the black; while the black were equal to the fourth part once more of the dappled and with it a fifth part, when all, including the bulls, went to pasture together. Now the dappled in four parts were equal in number to a fifth part and a sixth of the yellow herd. Finally the yellow were in number equal to a sixth part and a seventh of the white herd. If thou canst accurately tell, O stranger, the number of cattle of the Sun, giving separately the number of well-fed bulls and again the number of females according to each colour, thou wouldst not be called unskilled or ignorant of numbers, but not yet shalt thou be numbered among the wise.

The Cattle Problem, Pt. II

We let w, b, d, and y denote the number of white (resp. black, dappled, yellow) cows. Then

$$\begin{aligned} w &= (\tfrac{1}{3}+\tfrac{1}{4})(B+b), & b &= (\tfrac{1}{4}+\tfrac{1}{5})(D+d) \\ d &= (\tfrac{1}{5}+\tfrac{1}{6})(Y+y), & y &= (\tfrac{1}{6}+\tfrac{1}{7})(W+w). \end{aligned}$$

Solving this system is no harder than before, but the numbers are more irritating and the fundamental solution is therefore larger. We find

$$293391w = 176400B + 79380D + 29106Y + 9009W,$$

and our solution from Part I now gives $w = \frac{7206360}{4657}t$. Thus

$$\begin{array}{llll} w = 7206360s & d = 3515820s & W = 10366482s & D = 7358060s \\ y = 5439213s & b = 4893246s & Y = 4149387s & B = 7460514s, \end{array}$$

for $s \in \mathbb{Z}$.

The Cattle Problem, Pt. III

And here's where it starts to get interesting:

> But come, understand also all these conditions regarding the cattle of the Sun. When the white bulls mingled their number with the black, they stood firm, equal in depth and breadth, and the plains of Thrinacia, stretching far in all ways, were filled with their multitude. Again, when the yellow and the dappled bulls were gathered into one herd they stood in such a manner that their number, beginning from one, grew slowly greater till it completed a triangular figure, there being no bulls of other colours in their midst nor none of them lacking. If thou art able, O stranger, to find out all these things and gather them together in your mind, giving all the relations, thou shalt depart crowned with glory and knowing that thou hast been adjudged perfect in this species of wisdom.

The Cattle Problem, Pt. III

In other words, $W + B$ is a square number, while $Y + D$ is triangular.

1. Since $W + B = 2^2 \cdot 3 \cdot 11 \cdot 29 \cdot 4657s$, choosing $s = 3 \cdot 11 \cdot 29 \cdot 4657 \cdot n^2$ gives the first condition.
2. The second is harder. We need

$$Y + D = 3 \cdot 7 \cdot 11 \cdot 29 \cdot 353 \cdot 4657^2 \cdot n^2 = \frac{k(k+1)}{2}.$$

for some $k \in \mathbb{Z}$.

Completing the square in k gives

$$2^3 \cdot 7 \cdot 11 \cdot 29 \cdot 353 \cdot 4657^2 \cdot n^2 = (2k+1)^2 - 1.$$

Now, defining $m = 2k + 1$, we write

$$m^2 - Cn^2 = 1,$$

in which $C = 4729494 \cdot 9314^2$. What remains is a Pell Equation.

Pell's Equation

Pell's Equation

Pell's equation is a name for any Diophantine equation of the form

$$x^2 - ny^2 = 1,$$

in which n is a given positive, non-square integer.[2]

Pell's Equation (in the special case $n = 2$) was first considered around 400 BC in India and Greece for its connection to $\sqrt{2}$. The approximation $x \approx \sqrt{n}y$ holds when x, y are large, as was known to the Pythagoreans.

Archimedes once gives the approximation $\sqrt{3} = \frac{1351}{780}$, which (while unexplained in his work) satisfies $1351^2 - 3 \cdot 780^2 = 1$.

[2] In other words, points on a hyperbola with integer coordinates.

Pell's Equation in India

The first major step towards a systematic treatment of the Pell Equation came from the Indian mathematician Brahmagupta, who showed that

$$(x_1^2 - Ny_1^2)(x_2^2 - Ny_2^2) = (x_1x_2 + Ny_1y_2)^2 - N(x_1y_2 + x_2y_1)^2.$$

($\Longrightarrow$ numbers of the form $a^2 + Nb^2$ are closed under multiplication.)

This allowed Brahmagupta to compose solutions to the Pell Equation, and in particular, show the existence of infinitely many solutions. His method was furthered by Bhaskara II in 1150.

Pell's Equation in Europe

Progress in India escaped the notice of European mathematicians, who would not discover these results until the 17th century.

(1657) Pierre de Fermat finds a solution and solves $x^2 - Ny^2$ for $N \leq 150$. Challenges to address other cases circulate to Wallis and Broucnker.

(1659) Brouncker's solution is discussed in Rahn's *Teutsche Algebra*. This is translated into English by Branker and revised by John Pell. Euler attributes the work to Pell by mistake.

(1766) A modern theory of Pell's Equation based on continued fractions is developed by Lagrange.

$$\vdots$$

(1880) A. Amthor solves the Cattle Problem, but can only approximate the total as $\approx 7.76 \times 10^{206544}$.

Pell's Equation via Continued Fractions

Continued Fractions

Brahmagupta's method to compose solutions can be used to find many solutions to Pell's Equation **only when a first solution is known.**

A classic take on this problem finds an initial solution using the theory of continued fractions, expressions of the form

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots}}}}$$

To save space, we write these in the form $[a_0, a_1, a_2, \ldots]$.

An Algorithm for Continued Fractions

A continued fraction for the positive real number x may be computed as follows.

1. Define $a_0 = \lfloor x \rfloor$, the greatest integer less than or equal to x.
2. If $a_0 = x$, stop. Otherwise, replace x by $1/(x - a_0)$ and return to (1) to find a_1, etc.

Example: Compute the continued fraction for $\frac{49}{38}$.

$$a_0 = \lfloor \tfrac{49}{38} \rfloor = 1 \qquad a_1 = \lfloor 1/(\tfrac{49}{38} - 1) \rfloor = \lfloor \tfrac{38}{11} \rfloor = 3$$
$$a_2 = \lfloor 1/(\tfrac{38}{11} - 3 \rfloor = \lfloor \tfrac{11}{5} \rfloor = 2 \qquad a_3 = \lfloor 1/(\tfrac{11}{5} - 2) \rfloor = \lfloor 5 \rfloor = 5$$

Thus $\frac{49}{38} = [1, 3, 2, 5]$.

Continued Fractions and Pell's Equation

Given a real number $x = [a_0, a_1, \ldots,]$, we define the nth convergent to be the finite continued fraction $[a_0, \ldots, a_n]$. Convergents are rational.

Theorem (Legendre 1768):

The Pell Equation $x^2 - ny^2 = 1$ has infinitely many solutions. The solution that minimizes $|x|$ appears as a convergent in the continued fraction expansion of $\sqrt{n}$.

As an example, note that $\sqrt{7} = [2, \overline{1, 1, 1, 4}]$. The convergents to $\sqrt{7}$ are

$$[2] = 2, \quad [2, 1] = 3, \quad [2, 1, 1] = \tfrac{5}{2}, \quad [2, 1, 1, 1] = \tfrac{8}{3}, \ldots$$

and this last convergent corresponds to the solution $8^2 - 7 \cdot 3^2 = 1$.

Back to the Cattle Problem

Recall that the Cattle Problem corresponds to the Pell Equation

$$m^2 - Cn^2 = 1,$$

in which $C = 4729494 \cdot 9314^2$. We compute

$$\sqrt{C} = [6398085, 29, 5, 1, 2, 3, 1, 3, 1, 1, 54, 4, \ldots].$$

and find (after a lot[3] of computation) a convergent m/n which satisfies $m^2 - Cn^2 = 1$. This has $n \approx 1.86 \times 10^{103265}$.

We conclude that the herd consists of $\approx 7.76 \times 10^{206544}$ cattle.

[3]On the order of $100,000$ convergents.

Amthor's Solution

Amthor's original solution to the Cattle Problem differed slightly from the idealistic version I presented before. To make the computation tractable, Amthor first solved

$$m^2 - C'n^2 = 1,$$

in which $C' = 4729494$, the square-free part of our full C. While $\sqrt{C}$ has a continued fraction of tens of thousands of terms, $\sqrt{C'}$ is easy:

$$\begin{aligned}\sqrt{C'} = [2174, &\overline{1, 2, 1, 5, 2, 25, 3, 1, 1, 1, 1, 1, 1, 15, 1, 2, 16, 1, 2, 1, 1, 8, 6, 1,}\\ &\overline{21, 1, 1, 3, 1, 1, 1, 2, 2, 6, 1, 1, 5, 1, 17, 1, 1, 47, 3, 1, 1, 6, 1, 1, 3, 47,}\\ &\overline{1, 1, 17, 1, 5, 1, 1, 6, 2, 2, 1, 1, 1, 3, 1, 1, 21, 1, 6, 8, 1, 1, 2, 1, 16, 2, 1,}\\ &\overline{15, 1, 1, 1, 1, 1, 1, 3, 25, 2, 5, 1, 2, 1, 4348}]\end{aligned}$$

It takes only 92 convergents to first solve $m^2 - C'n^2 = 1$. From here, we compose solutions until $9314 \mid n$.

Further Context for the Cattle Problem

Archimedes and Apollonius

The exact purpose of the Cattle Problem remains unclear.[4] One possible explanation comes from the inferred rivalry between Archimedes and another mathematician, Apollonius of Perga (c. 262-192 BC).

This theory is supported by two incidents of one-upsmanship:

1. Per Eutocius, Apollonius wrote a book called *Easy Delivery*[5] which improved upon Archimedes' bounds

$$3 + \frac{10}{71} < \pi < 3 + \frac{10}{70}$$

from *Measurement of the Circle*.

[4] Frankly, the authorship is also debatable.

[5] Or the more suggestive translation, *Quick Bringing-to-Birth*.

Archimedes and Apollonius

2. Per the *Collection* of Pappus (c. 290-350 AD), Apollonius wrote a book concerning the representation and multiplication of large numbers. This can be seen as a reply to Archimedes' *The Sand Reckoner*, in which Archimedes more-or-less invents a full base-10 positional system, discovers the law of exponents, and estimates the size of the universe.

The Cattle Problem may thus be a response to the computational works of Apollonius. Archimedes has imagined a problem for which he believes

a. has a solution
b. requires computations large enough to be infeasible

Note – This type of problem-solving contest between mathematicians was common until the Renaissance.

Questions?