Representation via Egyptian Fractions

Problems from the History of Mathematics

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Brown University

Egyptian Fractions

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An Egyptian fraction is a representation of a rational number p/q as a sum of distinct unit fractions. For example,

$$\frac{15}{36} = \frac{1}{3} + \frac{1}{12} = \frac{1}{4} + \frac{1}{6}.$$

Egyptian fractions were the standard means to represent rational numbers in ancient Egypt, and their use continued into early European mathematics.

While it is not known why the Egyptians used this system, two probable causes are

1. Ease of division of goods into equal parts. (To split 15 pizzas among 36 people, split 9 pizzas into quarters and 6 into sixths.)
2. Division as a concept inspired by the study of reciprocals of integers.

The Rhind Papyrus

The Rhind papyrus is an Egyptian mathematical papyrus dated to c. 1650 BC, although parts were copied from earlier texts dated to c. 1850 BC.

It begins with a table known as the $2/n$ table, which lists Egyptian fractions for the numbers $2/n$ with $n \leq 101$ odd.

n	$\frac{2}{n}$	$1+\alpha$ [1]
3	$\overline{2} + \overline{6}$	$1 + \overline{2}$
5	$\overline{3} + \overline{15}$	$1 + \overline{3}$
7	$\overline{4} + \overline{28}$	$1 + \overline{2} + \overline{4}$
9	$\overline{6} + \overline{18}$	$1 + \overline{2}$
11	$\overline{6} + \overline{66}$	$1 + \bar{\bar{3}} + \overline{6}$
13	$\overline{8} + \overline{52} + \overline{104}$	$1 + \overline{2} + \overline{8}$
15	$\overline{10} + \overline{30}$	$1 + \overline{2}$
17	$\overline{12} + \overline{51} + \overline{68}$	$1 + \overline{3} + \overline{12}$
19	$\overline{12} + \overline{76} + \overline{114}$	$1 + \overline{2} + \overline{12}$
21	$\overline{14} + \overline{42}$	$1 + \overline{2}$
23	$\overline{12} + \overline{276}$	$1 + \bar{\bar{3}} + \overline{4}$
25	$\overline{15} + \overline{75}$	$1 + \bar{\bar{3}}$
27	$\overline{18} + \overline{54}$	$1 + \overline{2}$
29	$\overline{24} + \overline{58} + \overline{174} + \overline{232}$	$1 + \overline{6} + \overline{24}$
31	$\overline{20} + \overline{124} + \overline{155}$	$1 + \overline{2} + \overline{20}$
33	$\overline{22} + \overline{66}$	$1 + \overline{2}$
35	$\overline{30} + \overline{42}$	$1 + \overline{6}$
37	$\overline{24} + \overline{111} + \overline{296}$	$1 + \overline{2} + \overline{24}$
39	$\overline{26} + \overline{78}$	$1 + \overline{2}$
41	$\overline{24} + \overline{246} + \overline{328}$	$1 + \bar{\bar{3}} + \overline{24}$
43	$\overline{42} + \overline{86} + \overline{129} + \overline{301}$	$1 + \overline{42}$
45	$\overline{30} + \overline{90}$	$1 + \overline{2}$
47	$\overline{30} + \overline{141} + \overline{470}$	$1 + \overline{2} + \overline{15}$
49	$\overline{28} + \overline{196}$	$1 + \overline{2} + \overline{4}$
51	$\overline{34} + \overline{102}$	$1 + \overline{2}$

n	$\frac{2}{n}$	$1+\alpha$ [1]
53	$\overline{30} + \overline{318} + \overline{795}$	$1 + \bar{\bar{3}} + \overline{10}$
55	$\overline{30} + \overline{330}$	$1 + \bar{\bar{3}} + \overline{6}$
57	$\overline{38} + \overline{114}$	$1 + \overline{2}$
59	$\overline{36} + \overline{236} + \overline{531}$	$1 + \overline{2} + \overline{12} + \overline{18}$
61	$\overline{40} + \overline{244} + \overline{488} + \overline{610}$	$1 + \overline{2} + \overline{40}$
63	$\overline{42} + \overline{126}$	$1 + \overline{2}$
65	$\overline{39} + \overline{195}$	$1 + \bar{\bar{3}}$
67	$\overline{40} + \overline{335} + \overline{736}$	$1 + \overline{2} + \overline{8} + \overline{20}$
69	$\overline{46} + \overline{138}$	$1 + \overline{2}$
71	$\overline{40} + \overline{568} + \overline{710}$	$1 + \overline{2} + \overline{4} + \overline{40}$
73	$\overline{60} + \overline{219} + \overline{292} + \overline{365}$	$1 + \overline{6} + \overline{20}$
75	$\overline{50} + \overline{150}$	$1 + \overline{2}$
77	$\overline{44} + \overline{308}$	$1 + \overline{2} + \overline{4}$
79	$\overline{60} + \overline{237} + \overline{316} + \overline{790}$	$1 + \overline{4} + \overline{15}$
81	$\overline{54} + \overline{162}$	$1 + \overline{2}$
83	$\overline{60} + \overline{332} + \overline{415} + \overline{498}$	$1 + \overline{3} + \overline{20}$
85	$\overline{51} + \overline{255}$	$1 + \bar{\bar{3}}$
87	$\overline{58} + \overline{174}$	$1 + \overline{2}$
89	$\overline{60} + \overline{356} + \overline{534} + \overline{890}$	$1 + \overline{3} + \overline{10} + \overline{20}$
91	$\overline{70} + \overline{130}$	$1 + \overline{5} + \overline{10}$
93	$\overline{62} + \overline{186}$	$1 + \overline{2}$
95	$\overline{60} + \overline{380} + \overline{570}$	$1 + \overline{2} + \overline{12}$
97	$\overline{56} + \overline{679} + \overline{776}$	$1 + \overline{2} + \overline{8} + \overline{14} + \overline{28}$
99	$\overline{66} + \overline{198}$	$1 + \overline{2}$
101	$\overline{101} + \overline{202} + \overline{303} + \overline{606}$	1

The Rhind Papyrus

There is some debate as to how Ahmes and other Egyptian scribes would have prepared such tables. The last entry in the $2/n$ table suggests that

$$\frac{2}{n} = \frac{1}{n} + \frac{1}{2n} + \frac{1}{3n} + \frac{1}{6n}$$

was known, but fewer terms and smaller denominators are preferred.

For general p/q with q composite, the Egyptians would attempt to write p as a sum of divisors of q. This is not always possible (eg. $5/21$). What survives is likely a mix of formulas and ad hoc results.

A Greedy Algorithm for Egyptian Fractions

It is not clear that every rational number even has an Egyptian fraction. The first proof of this result is due to Leonardo of Pisa (Fibonacci) and appears in his *Liber Abaci* in 1202. Fibonacci's proof is an example of a greedy algorithm:

1. Given $p/q < 1$, let n_1 be minimal such that $\frac{1}{n_1} \leq \frac{p}{q} \quad \left(< \frac{1}{n_1 - 1}\right)$.
2. Define
$$\frac{p}{q} - \frac{1}{n_1} = \frac{n_1 p - q}{q n_1} =: \frac{p_1}{q_1}.$$
3. Repeat from (1) with p_1/q_1 in place of p/q. This process will terminate because $p_1 < p$.

The solutions this produces are not always very elegant. For example,

$$\begin{aligned}\frac{29}{61} &= \frac{1}{3} + \frac{1}{8} + \frac{1}{59} + \frac{1}{7853} + \frac{1}{96901533} + \frac{1}{21909783131182008} \\ &= \frac{1}{3} + \frac{1}{8} + \frac{1}{60} + \frac{1}{2440}.\end{aligned}$$

Open Problems in Egyptian Fractions

Egyptian fractions are of continued interest to number theorists today. Some open problems include the following:

1. It is known that each proper fraction with denominator q has an Egyptian fraction of length $O(\sqrt{\log q})$. A conjecture of Erdős claims $O(\log \log q)$ suffices.
2. The Erdős–Strauss Conjecture:

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

   has a solution for each $n \geq 1$.

Questions?